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The effect of overtaking disturbances on the motion of converging shock waves

By M. YOUSAF
Mathematics Department, Laurentian University, Sudbury, Ontario

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In this paper an exact formulation of the strength of the disturbance overtaking a shock is presented. The similarity solution is used to find the five interaction terms at all points of the flow. This work confirms that when the strength of the overtaking disturbance is known the CCW approximation may be modified to become an exact theory.

1. Introduction

The motion of a shock wave down a channel has been studied by Chester (1954, 1960), Chisnell (1957) and Whitham (1957, 1958). Chester considered the motion of a shock through a channel consisting of a section of slowly varying area separating two sections of uniform area. By linearizing the problem with respect to the area difference between the uniform sections, he obtained a complete description of the flow. In particular, he showed that when the shock was well past the area change it became a uniform shock whose Mach number differed from the incident shock Mach number M by an amount δM given by

$$\frac{\delta A}{A} = -\frac{2M\,\delta M}{(M^2-1)\,K(M)}, \tag{1.1}$$

in terms of the area change of the channel, where $K(M)$ is a monotonic function of M varying from 0·5 for weak shocks to $K_\infty = 0{\cdot}394\ldots$ for strong shocks.

Chisnell integrated Chester's result and obtained a closed-form expression for $A(M)$. He suggested that this expression be used to give an approximate description of the motion of a shock down a channel containing continuous finite area changes. In Chester's problem no disturbances exist in the flow behind the incident shock; hence Chisnell's description is that of a freely propagating shock, i.e. a shock not affected by disturbances in the flow behind the shock. Whitham obtained the same $A(M)$ expression by the application of a characteristic rule. He applied the differential relation valid along the characteristics which overtake the shock to the flow variables immediately behind the shock, which are given by the Rankine–Hugoniot relations. In this paper the overtaking characteristics are the C_- ones and the differential relation valid along C_- is

$$\frac{1}{\rho a}dp - du + \frac{ua}{u-a}\frac{dA}{A} = 0. \tag{1.2}$$

Following Hayes (1968), a shock motion given by this $A(M)$ expression will be referred to as a CCW description in this paper.

Rościszewski (1960) and Oshima *et al.* (1965) have formulated the error involved in using the CCW description. They achieved this by integrating (1.2) along two neighbouring overtaking characteristics and forming the difference of the two integrals.

In this paper an exact formulation of the strength of the disturbance overtaking the shock is presented. This is obtained by applying the differential relations valid along characteristics to an elementary quadrilateral composed of C_+ and C_- characteristics. Changes in flow quantities along an edge of the quadrilateral are decomposed into contributions arising from crossing C_+ and C_- disturbances, elementary contact discontinuities, and area changes. An exact equation is derived from the characteristic equation which describes the variation in the velocity jump across the elementary overtaking C_- disturbance during its passage through the elementary quadrilateral. The variation in the change in velocity across the overtaking disturbance is found to consist of five terms. These are interpreted as due to interactions between any two of the C_+ and C_- disturbances, contact discontinuities and area changes, with the exception that there is no contribution from the C_+, C_- interaction. Following the idea of Rościszewski and Oshima, this result is integrated along an overtaking characteristic and gives the strength of the disturbance that overtakes the shock during its passage between the two neighbouring overtaking C_- characteristics. An analysis of an elementary triangle composed of the shock path and a C_+ and a C_- characteristic shows the effect of the overtaking disturbance on the path of the shock. This analysis, incidentally, demonstrates the equivalence of Whitham's derivation of $A(M)$ by applying (1.2) along the shock path and Chisnell's approach of neglecting overtaking disturbances. The exact description of the shock motion is then expressed in the form

$$\frac{\delta A}{A}(1+\lambda) = \frac{-2M\delta M}{(M^2-1)K(M)}, \tag{1.3}$$

where λ is an integral containing in its integrand the five interaction terms. This formula shows how the CCW description of (1.1) may be corrected by taking into account the overtaking disturbance.

The first application of the CCW approximation was to converging cylindrical and spherical shocks near the axis or centre of collapse. As a shock approaches the axis or centre of collapse, the strong shock equations are valid and the similarity solution of Guderley (1942), Butler (1954) and Stanyukovich (1960, p. 521) is available for comparison. The dependence of the shock Mach number M on the area A of the front is given by the CCW method from (1.1) as

$$M \propto A^{-\frac{1}{2}K_\infty}. \tag{1.4}$$

Chisnell gave the following table comparing the values of K_∞ from Butler's similarity results with his own work.

The surprisingly good agreement between the two theories has been discussed

	K_∞ for cylindrical shock		K_∞ for spherical shock	
γ	Chisnell	Butler	Chisnell	Butler
1·4	0·197070	0·197294	0·394141	0·394364
1·2	0·163112	0·161220	0·326223	0·320752
5/3	0·225425	0·226054	0·452108	0·452692

TABLE 1

by Chisnell and Whitham. Chisnell made an attempt to estimate directly the strength of the overtaking wave by considering only three of the five interactions occurring in the flow behind the shock. He found that the interaction of a C_+ with an area change produced only a small effect in comparison with the interaction of a C_- with an area change and with the interaction of a C_- with a contact discontinuity. Further, the two large interaction terms together had only a small effect. This fortuitous cancellation explained the remarkable agreement of the CCW approximation with the similarity solution.

In this paper the exact formulation containing five interaction terms is applied to this problem. The similarity solution is used to find the five interaction terms at all points of the flow. This work supplements Chisnell's work and confirms that there are two large, nearly equal and opposite interactions. In all cases that have been computed it is found that the three small interaction terms remain less than $\frac{1}{5}$ of each of the two large terms.

Table 1 shows that the agreement between the CCW and similarity solution is closest when $\gamma = 1{\cdot}4$. The numerical solutions of this paper explain this point. For $\gamma = 1{\cdot}2$ the sum of the two large interaction terms is positive and for $\gamma = \frac{5}{3}$ it is negative at all points of the flow. For $\gamma = 1{\cdot}4$ this sum is found to be positive near the shock and negative far from the shock. Hence interactions produced far from the shock tend to cancel those produced near the shock in the integral denoted by λ in (1.3).

The similarity solution has been computed again to obtain values of K correct to 13 figures and the effect of the overtaking wave has been computed. Substitution of the value of λ in (1.3) gives a revised exponent in (1.4), which in all cases agrees to 13 figures with the similarity solution. This work confirms that when the strength of the overtaking wave is known the CCW approximation may be modified to give an exact theory. Of course, the difficulty of estimating the strength of the overtaking wave in other problems remains.

2. Theory

In this section the disturbances which overtake a shock in unsteady one-dimensional flow in a channel are examined. The exact equations are used to show that changes in the jump in fluid velocity u across a C_- characteristic are produced by five different mechanisms. This is achieved by considering an elementary quadrilateral of C_+ and C_- characteristics. Each of the five mechanisms is identified as an interaction between two particular types of disturbance. The system of equations is

$$\frac{\partial p}{\partial \eta}+\rho a\frac{\partial u}{\partial \eta}+\frac{\rho a^2 u}{u+a}\frac{\partial(\log A)}{\partial \eta}=0 \quad \text{along } C_+ \text{ characteristics,} \tag{2.1}$$

$$\frac{\partial p}{\partial s}-\rho a\frac{\partial u}{\partial s}+\frac{\rho a^2 u}{u-a}\frac{\partial(\log A)}{\partial s}=0 \quad \text{along } C_- \text{ characteristics,} \tag{2.2}$$

where
$$\frac{\partial}{\partial \eta}=\frac{\partial}{\partial t}+(u+a)\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial s}=\frac{\partial}{\partial t}+(u-a)\frac{\partial}{\partial x}, \tag{2.3}$$

a denotes the velocity of sound, p the pressure, ρ the density, u the flow velocity and A the cross-sectional area of the tube

To obtain an equation for u, use is made of the commutative relation

$$\frac{\partial}{\partial s}\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \eta}\frac{\partial}{\partial s}=F\frac{\partial}{\partial \eta}-F\frac{\partial}{\partial s},$$

where
$$F=\frac{1}{2a}\left[\frac{\partial}{\partial s}(u+a)-\frac{\partial}{\partial \eta}(u-a)\right],$$

which follows directly from (2.3). By operating on (2.1) with $\delta s(\partial/\partial s-F)\,\delta\eta$ and on (2.2) with $\delta\eta(\partial/\partial\eta-F)\,\delta s$ and subtracting, the pressure p is eliminated to get the following equation for an elementary C_+, C_- quadrilateral:

$$\begin{aligned}2\rho a\frac{\partial^2 u}{\partial s\,\partial\eta}+\frac{\partial u}{\partial \eta}\left[\rho\frac{\partial a}{\partial s}+a\frac{\partial \rho}{\partial s}\right]+\frac{\partial u}{\partial s}\left[\rho\frac{\partial a}{\partial \eta}+a\frac{\partial \rho}{\partial \eta}\right]&\\ +\frac{\partial}{\partial s}\left[\frac{\rho a^2 u}{u+a}\right]\frac{\partial(\log A)}{\partial \eta}-\frac{\partial}{\partial \eta}\left[\frac{\rho a^2 u}{u-a}\right]\frac{\partial(\log A)}{\partial s}-2\rho a\frac{\partial u}{\partial \eta}F&\\ -\frac{2\rho a^3 u}{u^2-a^2}\frac{\partial^2(\log A)}{\partial s\,\partial\eta}+\frac{2\rho a^3 u}{u^2-a^2}F\frac{\partial(\log A)}{\partial \eta}&=0.\end{aligned} \tag{2.4}$$

The exact characteristic equations (2.1) and (2.2) describe changes in flow quantities along edges of this elementary quadrilateral composed of C_+ and C_- characteristics. Changes in flow quantities along an edge of the quadrilateral are now decomposed into contributions arising from crossing C_+ and C_- disturbances, elementary contact discontinuities and area changes. The jumps in flow quantities across a C_+ characteristic are related by

$$dp_+=\rho a\,du_+, \quad d\rho_+=(\rho/a)\,du_+, \quad da_+=\tfrac{1}{2}(\gamma-1)\,du_+, \tag{2.5}$$

and those across a C_- disturbance by

$$dp_-=-\rho a\,du_-, \quad d\rho_-=-(\rho/a)\,du_-, \quad da_-=-\tfrac{1}{2}(\gamma-1)\,du_-. \tag{2.6}$$

The changes in flow quantities through an area change are given by the steady channel flow equations

$$du_A=-\frac{a^2 u}{a^2-u^2}\frac{dA}{A}, \quad dp_A=\frac{\gamma p u^2}{a^2-u^2}\frac{dA}{A}, \tag{2.7a}$$

$$d\rho_A=\frac{\rho u^2}{a^2-u^2}\frac{dA}{A}, \quad da_A=\frac{(\gamma-1)\,a u^2}{2(a^2-u^2)}\frac{dA}{A}. \tag{2.7b}$$

Along a C_+-characteristic side of the quadrilateral changes in u can arise from crossing C_- disturbances and area changes, hence

$$\frac{\partial u}{\partial \eta} = \frac{\partial u_-}{\partial \eta} + \frac{\partial u_A}{\partial \eta}. \tag{2.8}$$

Contact discontinuities crossing this C_+ side produce changes in ρ and a, but not in u, hence

$$\frac{\partial \rho}{\partial \eta} = \frac{\partial \rho_c}{\partial \eta} + \frac{\partial \rho_-}{\partial \eta} + \frac{\partial \rho_A}{\partial \eta}, \quad \frac{\partial a}{\partial \eta} = \frac{\partial a_c}{\partial \eta} + \frac{\partial a_-}{\partial \eta} + \frac{\partial a_A}{\partial \eta}. \tag{2.9}$$

Similarly for a C_- side, we have

$$\left.\begin{aligned} \frac{\partial u}{\partial s} &= \frac{\partial u_+}{\partial s} + \frac{\partial u_A}{\partial s}, \quad \frac{\partial \rho}{\partial s} = \frac{\partial \rho_c}{\partial s} + \frac{\partial \rho_+}{\partial s} + \frac{\partial \rho_A}{\partial s}, \\ \frac{\partial a}{\partial s} &= \frac{\partial a_c}{\partial s} + \frac{\partial a_+}{\partial s} + \frac{\partial a_A}{\partial s}. \end{aligned}\right\} \tag{2.10a, b, c}$$

At a point in the flow where u, ρ, a and their derivatives with respect to η and s are assumed known, the decomposition into the various disturbances is achieved as follows. As A is a known function of x, $\partial u_A/\partial \eta$ and $\partial u_A/\partial s$ are given by (2.7), hence $\partial u_-/\partial \eta$ and $\partial u_+/\partial s$ follow from (2.8) and (2.10a), and then $\partial \rho_-/\partial \eta$, $\partial a_-/\partial \eta$, $\partial \rho_+/\partial s$ and $\partial a_+/\partial s$ are given by (2.5) and (2.6). Finally $\partial \rho_c/\partial \eta$, $\partial a_c/\partial \eta$, $\partial \rho_c/\partial s$ and $\partial a_c/\partial s$ are given by (2.9) and (2.10).

By considering the t co-ordinates of the vertices of the elementary quadrilateral $PQRS$ in figure 1 consisting of C_+ and C_- characteristics the following first-order relation is obtained:

$$\delta s\, \partial(\delta \eta)/\partial s = \partial \eta\, \partial(\delta s)/\partial \eta. \tag{2.11}$$

Consideration of the x co-ordinates of the vertices of the quadrilateral gives the following relation:

$$\delta \eta\, \partial\{(u-a)\,\delta s\}/\partial \eta = \delta s\, \partial\{(u+a)\,\delta \eta\}/\partial s. \tag{2.12}$$

Combining (2.11) and (2.12), one obtains

$$\delta s \frac{\partial}{\partial s}(\delta \eta) = \delta \eta \frac{\partial}{\partial \eta}(\delta s) = \frac{1}{2a}\left[\frac{\partial}{\partial \eta}(u-a) - \frac{\partial}{\partial s}(u+a)\right]\delta \eta\, \delta s = -F\,\delta \eta\, \delta s. \tag{2.13}$$

An equation for $\partial[(\partial u_-/\partial \eta)\,\delta \eta]/\partial s$ can now be obtained. Equation (2.13) shows that

$$\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial \eta}\delta \eta\right) = \frac{\partial^2 u}{\partial s\, \partial \eta}\delta \eta - F\frac{\partial u}{\partial \eta}\delta \eta,$$

and $\partial^2 u/\partial s\, \delta \eta$ is given by (2.4). Introducing the decomposition formulae (2.8)–(2.10) it follows after some simplification that

$$\begin{aligned} \frac{\partial}{\partial s}\left\{\frac{\partial u_-}{\partial \eta}\delta \eta\right\} &= -\frac{1}{4\rho}\frac{\partial u_+}{\partial s}\frac{\partial \rho_c}{\partial \eta}\delta \eta - \frac{1}{4\rho}\frac{\partial u_-}{\partial \eta}\frac{\partial \rho_c}{\partial s}\delta \eta \\ &\quad - \left[\frac{(\gamma-1)\,u^2 + 2a^2}{4(u-a)^2}\right]\frac{\partial u_-}{\partial \eta}\frac{\partial}{\partial s}(\log A)\,\delta \eta - \left[\frac{(\gamma-1)\,u^2 - 2a^2}{4(u-a)^2}\right]\frac{\partial u_+}{\partial s}\frac{\partial}{\partial \eta}(\log A)\,\delta \eta \\ &\quad + \left[\frac{\partial \rho_c}{\partial s}\frac{\partial}{\partial \eta}(\log A) - \frac{\partial \rho_c}{\partial \eta}\frac{\partial}{\partial s}(\log A)\right]\frac{a^2 u^2\, \delta \eta}{2\rho(a+u)\,(a-u)^2} = H\delta \eta. \end{aligned} \tag{2.14}$$

This equation describes the variation of the change in velocity across a C_- characteristic during its passage through the elementary quadrilateral com-

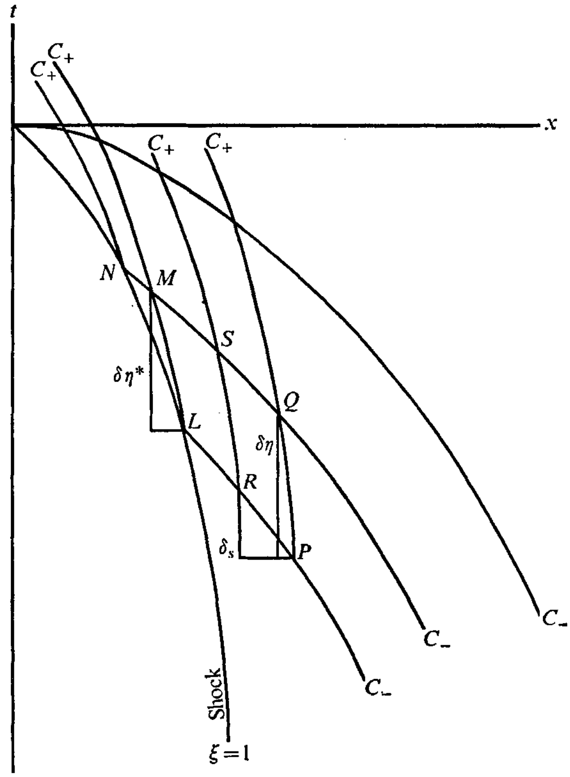

FIGURE 1. x, t diagram for the implosion of a shock wave. $\xi = 1$ is the shock-front line, $\xi = \xi_0$ is the ξ_0 line. Several characteristics of the C_+ and C_- families are also indicated.

posed of C_+ and C_- characteristics. The variation in the change of velocity across a C_- characteristic is due to the following five interaction terms.

(i) A change in velocity across a C_+ characteristic interacting with a contact discontinuity.

(ii) A change in velocity across a C_- characteristic interacting with a contact discontinuity.

(iii) A change in velocity across a C_- characteristic interacting with an area change.

(iv) A change in velocity across a C_+ characteristic interacting with an area change.

(v) A contact discontinuity interacting with an area change.

Integration of (2.14) with respect to s from $t = -\infty$ gives the strength of the overtaking disturbance at the last C_+, LM in figure 1, behind the shock as

$$\frac{\partial u_-}{\partial \eta}\delta\eta^* = \int_{-\infty}^{s^*} \{H\,\delta\eta\}\,ds, \tag{2.15}$$

where η^* and s^* are the values of η and s at the shock. The effect of the overtaking disturbance on the shock is now obtained by considering the elementary triangle LMN of figure 1. The C_- characteristic equation

$$\frac{dp}{\rho a} - du + \frac{au}{u-a}\frac{dA}{A} = 0 \tag{2.16}$$

is valid along MN. The increments dp, du and dA may be expressed in terms of the increments from M to L and L to N. As Whitham applied (2.16) to the shock front, we write

$$\left[\frac{dp}{\rho a} - du + \frac{au}{u-a}\frac{dA}{A}\right]_{\text{CCW}}$$

to describe the increment from L to N. The pressure increment from L to M is given by the characteristic equation

$$\frac{dp}{\rho a} + du + \frac{au}{u+a}\frac{dA}{A} = 0, \tag{2.17}$$

valid along C_+ lines, and the increment in u from L to M is $(\partial u/\partial \eta)\,\delta\eta^*$, given by (2.15). Hence

$$\left[\frac{dp}{\rho a} - du + \frac{au}{u-a}\frac{dA}{A}\right]_{\text{CCW}} + 2\frac{\partial u}{\partial \eta}\delta\eta^* + \frac{2ua^2}{u^2-a^2}\frac{dA}{A} = 0.$$

Use of (2.8) gives

$$\left[\frac{dp}{\rho a} - du + \frac{au}{u-a}\frac{dA}{A}\right]_{\text{CCW}} + 2\frac{\partial u_-}{\partial \eta}\delta\eta^* = 0. \tag{2.18}$$

This equation shows that the application of Whitham's characteristic rule and the neglect of overtaking C_- disturbances are equivalent. The $\delta\eta$ occurring in (2.15) may be related to $\delta\eta^*$, its value at the shock, by integrating (2.13) to give

$$\delta\eta = \delta\eta^* \exp\left\{\int_s^{s^*} F\,d\sigma\right\}. \tag{2.19}$$

As the slopes of LN and LM are U and $u+a$, $\delta\eta^*$ may be expressed in terms of δx by

$$\delta\eta^* = \left(\frac{U-u+a}{2aU}\right)\delta x. \tag{2.20}$$

Combining (2.15), (2.18), (2.19) and (2.20), we get

$$\left[\frac{dp}{\rho a} - du + \frac{au}{u-a}\frac{dA}{A}\right]_{\text{CCW}} + 2\left(\frac{U-u+a}{2aU}\right)dx\int_{-\infty}^{s^*}\left[\exp\left\{\int_s^{s^*} F\,d\sigma\right\}H\right]ds = 0. \tag{2.21}$$

Combining the area terms, we have

$$\left[\frac{dp}{\rho a} - du\right]\frac{u-a}{au} + \frac{dA}{A}(1+\lambda) = 0, \tag{2.22}$$

where
$$\lambda = \left(\frac{u-a}{a^2uU}\right)(U-u+a)A\frac{dx}{dA}\int_{-\infty}^{s^*}\exp\left\{\int_s^{s^*} F\,d\sigma\right\}H\,ds. \tag{2.23}$$

Thus we have derived an exact expression for λ, the correction to be applied to the CCW approximation for any channel flow. In the next section this correction to the CCW approximation will be evaluated for a particular problem.

3. Similarity solution

The exact similarly solution for converging cylindrical and spherical shocks near the point of collapse due to Guderley, Butler and Stanyukovich is now briefly described. Following Zel'dovich & Raizer (1967, p. 785), the system of equations for one-dimensional adiabatic flow of a perfect gas with constant specific heats with either cylindrical or spherical symmetry is written as

$$\left.\begin{aligned} \frac{\partial(\log \rho)}{\partial t}+u\frac{\partial(\log\rho)}{\partial r}+\frac{\partial u}{\partial r}+j\frac{u}{r} &= 0,\\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r} &= 0,\\ \frac{\partial}{\partial t}(\log p\rho^{-\gamma})+u\frac{\partial}{\partial r}(\log p\rho^{-\gamma}) &= 0,\end{aligned}\right\} \tag{3.1}$$

where $j = 1$ for the cylindrical case and $j = 2$ for the spherical case. In terms of the similarity variable $\xi = r/R$, where $R(t) = A(-t)^{\alpha}$ is the shock path, we write

$$p = \rho_0\frac{\xi^2R^2}{t^2}P(\xi), \quad \rho = \rho_0 G(\xi), \quad u = \frac{\xi R}{t}V(\xi). \tag{3.1a}$$

Substitution in (3.1) gives

$$\frac{dV}{d(\log\xi)} = \frac{\Delta_1(Z,V)}{\Delta(Z,V)},$$

$$\frac{d(\log G)}{d(\log\xi)} = \frac{\Delta_2(Z,V)}{\Delta(Z,V)}, \quad \frac{dZ}{d(\log\xi)} = \frac{\Delta_3(Z,V)}{\Delta(Z,V)}, \tag{3.2a,b}$$

where $$Z = \left(\frac{\gamma p}{G}\right)^{\frac{1}{2}}, \quad \Delta = \begin{vmatrix} 1 & V-\alpha & 0 \\ V-\alpha & Z/\gamma & 1/\gamma \\ 0 & (\gamma-1)Z & -1\end{vmatrix} = (V-\alpha)^2-Z$$

and the Δ_i ($i = 1, 2, 3$) are obtained by replacing the ith column of Δ by the column

$$\left(-(j+1)V, -\frac{2}{\gamma}Z-V(V-1), \quad 2\left(\frac{V-1}{V-\alpha}\right)Z\right).$$

Dividing (3.2c) by (3.2a), we obtain the first-order ordinary differential equation

$$dZ/dV = \Delta_3(Z,V)/\Delta_1(Z,V), \tag{3.3}$$

with boundary conditions

$$V(1) = \frac{2}{\gamma+1}\alpha, \quad Z(1) = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\alpha^2,$$

arising from the shock equations. In the next section the numerical procedure used to obtain the value of α to give a regular solution of (3.3) for the last overtaking C_- disturbances is described.

The five interaction terms given in (2.14) can be written in similarity form by using (2.5)–(2.10) and (3.1a) as

$$I_1 = \frac{1}{4\rho}\frac{\partial u_+}{\partial s}\frac{\partial \rho_c}{\partial \eta} = \frac{\xi R}{t^3}\frac{1}{4G}\left[(V-\alpha+Z^{\frac{1}{2}})\,\xi\, V' + V(V-1+Z^{\frac{1}{2}}) - \frac{jVZ}{V-Z^{\frac{1}{2}}}\right]$$
$$\times\left[(V-\alpha-Z^{\frac{1}{2}})\,\xi\left(G'-\frac{GV'}{Z^{\frac{1}{2}}}\right) - \frac{GV(V-1-Z^{\frac{1}{2}})}{Z^{\frac{1}{2}}} + jGV\right],$$

$$I_2 = \frac{1}{4\rho}\frac{\partial u_-}{\partial \eta}\frac{\partial \rho_c}{\partial s} = \frac{\xi R}{t^3}\frac{1}{4G}\left[(V-\alpha-Z^{\frac{1}{2}})\,\xi V' + V(V-1-Z^{\frac{1}{2}}) - \frac{jVZ}{V+Z^{\frac{1}{2}}}\right]$$
$$\times\left[(V-\alpha+Z^{\frac{1}{2}})\,\xi\left(G'+\frac{GV'}{Z^{\frac{1}{2}}}\right) + \frac{GV(V-1+Z^{\frac{1}{2}})}{Z^{\frac{1}{2}}} + jGV\right],$$

$$I_3 = \left[\frac{(\gamma-1)\,u^2+2a^2}{4(u-a)^2}\right]\frac{\partial u_-}{\partial \eta}\frac{\partial}{\partial s}(\log A) = \frac{j\xi R}{t^3}\left[\frac{(\gamma-1)\,V^2+2Z}{4(V+Z^{\frac{1}{2}})}\right]$$
$$\times\left[(V-\alpha-Z^{\frac{1}{2}})\,\xi V' + V(V-1-Z^{\frac{1}{2}}) - \frac{jVZ}{V+Z^{\frac{1}{2}}}\right],$$

$$I_4 = \left[\frac{(\gamma-1)\,u^2-2a^2}{4(u-a)^2}\right]\frac{\partial u_+}{\partial s}\frac{\partial}{\partial \eta}(\log A) = \frac{j\xi R}{t^3}\left[\frac{(\gamma-1)\,V^2-2Z}{4(V+Z^{\frac{1}{2}})^2}\right](V-Z^{\frac{1}{2}})$$
$$\times\left[(V-\alpha+Z^{\frac{1}{2}})\,\xi V' + V(V-1+Z^{\frac{1}{2}}) - \frac{jVZ}{V-Z^{\frac{1}{2}}}\right],$$

$$I_5 = \frac{a^2u^2}{2\rho(a-u)^2\,(a+u)}\left[\frac{\partial \rho_c}{\partial \eta}\frac{\partial}{\partial s}(\log A) - \frac{\partial \rho_c}{\partial s}\frac{\partial}{\partial \eta}(\log A)\right] = \frac{j\xi R}{t^3}\frac{V^2Z}{G(V-Z^{\frac{1}{2}})\,(V+Z^{\frac{1}{2}})^2}$$
$$\times\left[-\alpha Z^{\frac{1}{2}}\xi G' - \frac{G(V^2-\alpha V-Z)}{Z^{\frac{1}{2}}}\,\xi V' - \frac{GV^2(V-1)}{Z^{\frac{1}{2}}} + (j+1)\,GV Z^{\frac{1}{2}}\right].$$

The equation of a C_- characteristic $dr/dt = u-a$ may be written in similarity form as

$$dt/t = d\xi/\xi(V-\alpha+Z^{\frac{1}{2}}) \text{ by use of } (3.1a).$$

In similarity form $F = \nu(\xi)/t$, where

$$\nu(\xi) = \frac{1}{2Z^{\frac{1}{2}}}\left[\frac{(V-\alpha)\,\xi Z'}{Z^{\frac{1}{2}}} - 2Z^{\frac{1}{2}}(1+\xi V')\right].$$

Hence (2.19) may be written as

$$\delta\eta = \delta\eta^*\exp g, \tag{3.4}$$

where
$$g = \int_{\xi}^{\xi=1}\frac{\nu(\xi)\,d\xi}{\xi(V-\alpha+Z^{\frac{1}{2}})}. \tag{3.4a}$$

The similarity form of (2.23) is

$$\lambda = \frac{(V(1)+Z^{\frac{1}{2}}(1)\,(V(1)+Z^{\frac{1}{2}}(1)-\alpha)}{\alpha V(1)\,Z(1)\,j}\int_{\xi=1}^{\xi=\xi_0}\frac{T^{\alpha-2}f(\xi)\exp(g)\,d\xi}{V-\alpha+Z^{\frac{1}{2}}}, \tag{3.5}$$

where
$$T = \frac{t}{t^*},\quad f(\xi) = \frac{t^3}{\xi R}\sum_{i=1}^{5} I_i$$

and $\xi = \xi_0$ is the value of ξ on the limiting C_- characteristic. The computation of this expression for λ is discussed in the next section.

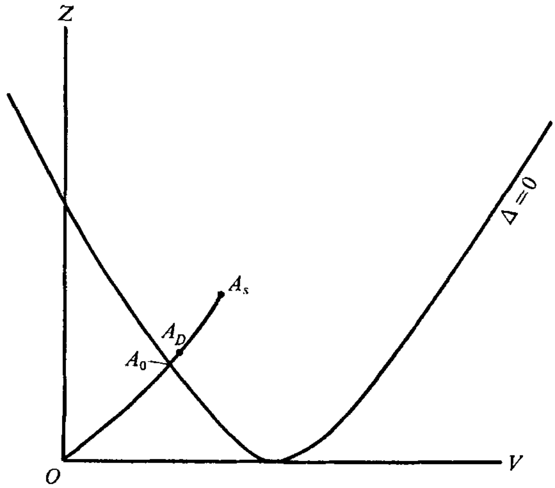

FIGURE 2. The integral curve starts at A_s, the shock point, passes through the saddle point A_0, where $\Delta = 0$, and ends at the origin, where $\xi = \infty$.

4. Numerical solution

To determine the parameter α occurring in (3.3), direct numerical integration is required. In the V, Z plane shown in figure 2, the shock point $\xi = 1$ is denoted by A_s and has co-ordinates $V = 2\alpha/(\gamma+1)$, $Z = 2\gamma(\gamma-1)\alpha^2/(\gamma+1)^2$, while $\xi = \infty$ is the origin. To obtain an integral curve from A_s to the origin, it is necessary to pass through a saddle point A_0. This saddle point is defined by $\Delta = 0 = \Delta_1$, and has $V = [w_1 + (w_1^2 - 4w_2)^{\frac{1}{2}}]/2\gamma j$, where $w_1 = \gamma[(j+1)\alpha - 1] + 2(1-\alpha)$ and $w_2 = 2\gamma\alpha(1-\alpha)j$. Zel'dovich has shown that $\Delta_2 = 0 = \Delta_3$ at A_0 also. A unique value of α ensures that the integral curve starting at A_s will arrive at A_0. Butler published the value of α correct to six figures and Welsh (1967) corrected the last decimal places in some cases.

In analysing the effect of the five different interaction terms in (2.14), greater accuracy is needed. The values of α have been found to 12 significant figures and the results are given in table 2. These were obtained by varying α and testing the value of Z at the end of the range of integration. In integrations from A_s to A_0 and from A_0 to A_s variations in the 13 significant figures of α were found to give values of Z at the end point correct to 13 figures. The integral

$$U = \int_{\xi_1}^{\xi_0} T^{\alpha-2} \frac{f(\xi)}{V-\alpha+Z^{\frac{1}{2}}} \exp(g)\, d\xi$$

occurring in (3.5) leads to

$$\frac{dU}{dV} = \frac{f(\xi)}{V-\alpha+Z^{\frac{1}{2}}} T^{\alpha-2} g\xi \frac{\Delta}{\Delta_1}.$$

The path of integration, a C_- characteristic, is defined by $dt/t = d\xi/\xi(V-\alpha+Z^{\frac{1}{2}})$ and may be written as

$$\frac{dT}{dV} = \frac{T}{V-\alpha+Z^{\frac{1}{2}}} \frac{\Delta}{\Delta_1} \quad \text{by use of (3.2}a\text{).} \tag{4.1}$$

γ	$j = 1$	$j = 2$
1·4	0·835 323 192 953	0·717 174 501 489
1·2	0·861 116 302 391	0·757 141 814 781
$\frac{5}{3}$	0·815 624 901 431	0·688 376 822 923

TABLE 2. Calculation of the similarity exponent α

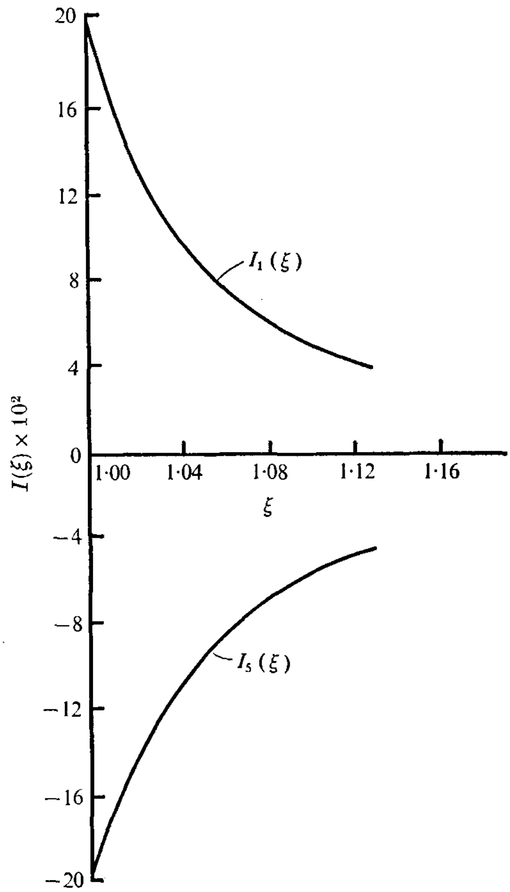

FIGURE 3. The two large interaction terms I_1 and I_5.

Equations (3.4a) and (3.2b) may be written as

$$\frac{dg}{dV} = \frac{\nu(\xi)}{V-\alpha+Z^{\frac{1}{2}}}\frac{\Delta}{\Delta_1}, \quad \frac{dG}{dV} = G\frac{\Delta_2}{\Delta_1}, \tag{4.2), (4.3}$$

by use of (3.2a).

Because of the singular behaviour of $T^{\alpha-2}$, a variable $W = \log(T^{\alpha-2}+g)$ is used in place of g, giving

$$\frac{dU}{dV} = \frac{f(\xi)}{V-\alpha+Z^{\frac{1}{2}}}\exp(W)\,\xi\frac{\Delta}{\Delta_1}. \tag{4.4}$$

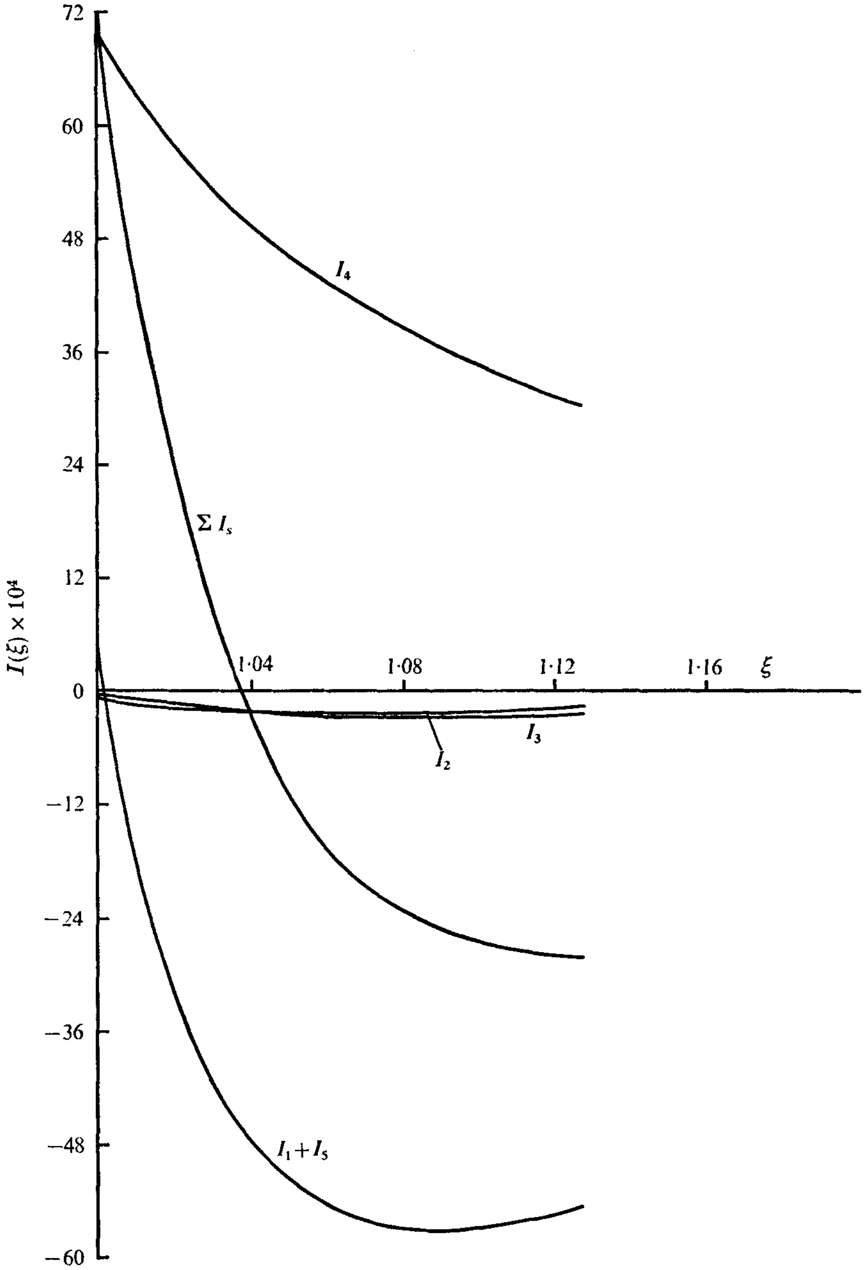

FIGURE 4. Sum of the two large interaction terms I_1+I_5, the three small interaction terms I_2, I_3 and I_4 and the sum of all the five interaction terms.

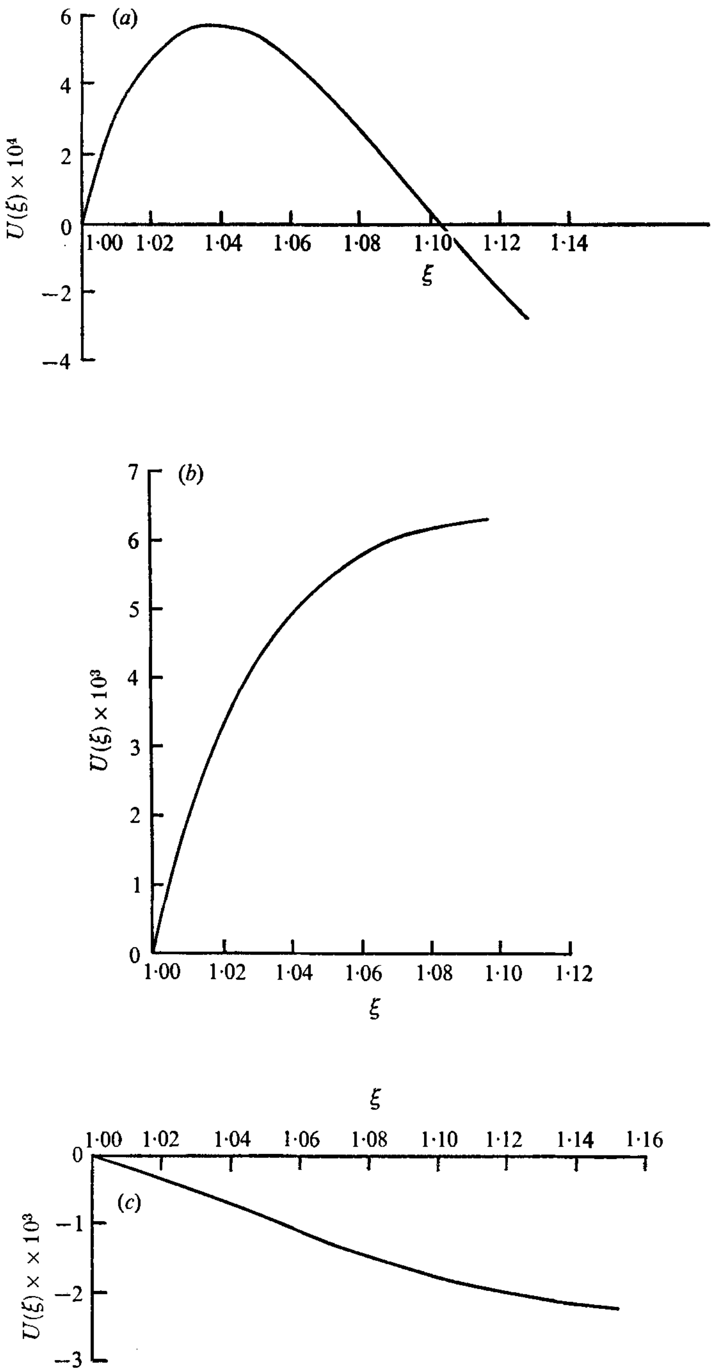

FIGURE 5. The integral U for the case $j = 2$ with (*a*) $\gamma = 1{\cdot}4$, (*b*) $\gamma = 1{\cdot}2$ and (*c*) $\gamma = \frac{5}{3}$.

γ	$j = 1$	$j = 2$
1·2	−0·011 595 591 367 297	−0·016 758 003 684 793
1·4	0·000 359 237 974 498	0·000 556 946 366 733
$\frac{5}{3}$	0·002 789 970 211 271	0·004 087 841 751 605

TABLE 3. Calculation of λ, the strength of the overtaking disturbances

Using (4.1) and (4.2), it follows that

$$\frac{dW}{dV} = \left[1 - \frac{(V-\alpha)\Delta_3}{2Z\Delta_1} + (\alpha-1)\frac{\Delta}{\Delta_1}\right]\frac{1}{V-\alpha+Z^{\frac{1}{2}}}. \tag{4.5}$$

Equations (4.1), (4.3), (4.4) and (4.5) together with (3.2a) and (3.3) are integrated from $\xi = 1$ to near A_0 using the correct value of α. The value of the integral from a point near A_0 to A_0 is evaluated using linear analysis. The values of λ are obtained for the six cases. This takes the CCW value of K from that in (1.1) to the exact value in (1.3), which agrees with Butler's K in all decimal places available.

The results for the case $j = 2$, $\gamma = 1{\cdot}4$ will now be discussed in detail. The cancellation of the five interaction terms is illustrated for this case in figures 3 and 4. In figure 3, the two big terms I_1 and I_5 are plotted against ξ, the similarity variable, and it is noted that I_1 is nearly the image of I_5 in the ξ axis. In figure 4, $I_1 + I_5$, the sum of the two large interaction terms, I_2, I_3 and I_4, the three small interaction terms, and the sum of all the interaction terms are plotted. It has been noted that the sum of the five interaction terms changes sign when ξ is between 1·03 and 1·04. It has also been noted that the sum $I_1 + I_5$ of the two large terms also changes sign near the shock. The further cancelling occurring in the case $\gamma = 1{\cdot}4$ is illustrated by the strength of the overtaking disturbance

$$U = \int_1^{\xi_0} \frac{T^{\alpha-2} f(\xi)\exp(g)}{V-\alpha+Z^{\frac{1}{2}}}\, d\xi$$

occurring in (3.5), which changes sign for $\gamma = 1{\cdot}4$ and $j = 2$ as shown in figure 5(a). This further cancellation give rise to close agreement between the CCW approximation and the similarity solution in the case $\gamma = 1{\cdot}4$. The above integral does not change sign in the other cases $\gamma = 1{\cdot}2$ and $\gamma = \frac{5}{3}$ as is shown in figures 5(b) and (c). Numerical solutions were also obtained for five other cases. In the case $\gamma = 1{\cdot}4$, $j = 1$, the sum of the two large interaction terms I_1 and I_5 again changes sign near the shock. The sum of the five interaction terms changes sign when ξ is between 1·05 and 1·06 and the integral occurring in (3.5) changes sign in this case. In the cases $\gamma = 1{\cdot}2, j = 1$ and 2, the sum of the two large interaction terms I_1 and I_5 remains positive and so do the sums of all five interaction terms and the integral in (3.5). In the cases $\gamma = \frac{5}{3}$, $j = 1$ and 2, the sum of two large interaction terms I_1 and I_5, the sum of all five interaction terms and the integral in (3.5) all remain negative. The values of λ occurring in (2.2), the strengths of the overtaking disturbances, were also calculated for the six cases and are given in table 3.

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